The Power of Relational Algebraic Thinking: a Study of Relational Thinking in Grade 8 in Banda Aceh

Mailizar[1]

[1]Dosen Program Studi Pendidikan Matematika, FKIP Unsyiah, Banda Aceh
Jalan T Nyak Arief, Kampus Unsyiah Darussalam Banda Aceh, 23211,
mailizar@fkip.unsyiah.ac.id

Abstract

Algebra is a gateway of technology improvement. It needs students to have good relational thinking in order for algebraic understanding to grow. Relational thinking means that the students develop the concept of a number that can vary, build upon similarities and differences, and appreciate for the equals sign as signifying equivalence of expressions. Three categories of relational thinking will be discussed in this paper: emerging, consolidating and established relational thinking. Analyzing data from some junior high school students in Banda Aceh it can be said that most students are still at the stage of emerging relational thinking. They construct limited responses to given questions. Many consolidating relational thinkers' students are concerned that the relationships hold only for a specific range of numbers. Established relational thinkers demonstrate good algebraic thinking processes. Three types of sentences have a potential for develop algebraic thinking with respect to equivalence, attention to operations, different numbers, compensation, and generalisation. Having presented students' responses to these three types of questions, we ask: how can teachers help students to move beyond the partial descriptions that characterize emerging relational thinkers.

Key words: Relational thinking, emergent, consolidating, and established

Introduction

Research on the development of algebraic thinking is urgently needed. According to Mathematical Association of America (2007), *Algebra: Gateway to a Technological Future*, it is said that "We need a much fuller picture of the essential early algebra ideas, how these ideas are connected to the existing curriculum, how they develop in children's thinking, how to scaffold this development, and what are the critical junctures of this development" (p.2). For this reason we need good problems in order to find out the development of the students' understanding of algebraic processes that can be used in solving different questions. The following problems could be a good for students to be solved and a good way for teachers to understand their thinking.

How might students think about this kind of problem? What numbers should it be in the Box? How do you find the missing numbers in these mathematical sentences?

$23 + 15 = 26 + \square$

$18 + \square = 20 + \square$

First we can expect that some students will employ purely computational methods to solve number sentences like the two given above. Our goal is to move students beyond purely arithmetic approaches to thinking about the kind of relationships that hold between the numbers. In the first number sentence, one number satisfies the relationship. In the second sentence, there are many possible solutions.

Students' Mathematical Thinking

Stephens (2007) mentioned that when using Computational Thinking, students recognize the field the problem belongs to firstly, and then activate the procedure they have already mastered to find the answer. In addition for instance, for the first problem above, a student might answer by having this kind of mathematical sentence.

$23 + 14 = 25 + \square \quad \rightarrow \quad 23 + 14 = 25 + \mathbf{12}$

How can the students find the answer? Having a good computational skill, a student might do like this:

a. *23 + 14 = 37*
37 – 25 = 12

b.
$$\begin{array}{r} 23 \\ 14 \\ \hline 37 \end{array} +$$
$$\begin{array}{r} 25 \\ \hline 12 \end{array} -$$

Another solution would be like the following. Since the relation between 23 in the first field and the 25 in the second field is 2 more, then there should be a relation between 14 and the number in the box that is 2 less. So the number in the box must be: 14 – 2 = 12. This kind of thinking is called ***a relational thinking*** of mathematics. The following picture illustrates the relational thinking process as mentioned above.

$$\overset{+2}{}$$

$$23 + 14 = 25 + \square$$

$$\underset{-2}{}$$

According to Molina, M., Castro, E., & Mason, J. (2008), students consider the number sentence as a whole in their mind, then analyze and find the structure and important elements or relationship to generate productive solutions. Other research from Carpenter (2001) and Stephens (2007, 2008) refer to relational thinking in the same way, when students see the equals sign as a relational symbol, students can focus on the structure of expression, and students carry out reasonable strategies to solve the number sentence attending to the operations involved.

A Study of Relational Thinking In Grade 8 In Banda Aceh

This study of relational thinking was conducted in Year 8 in Banda Aceh, Indonesia in a State Junior High School. For these older students, relational thinking needs to encompass all four operations. The questionnaire used two different types of number sentences/problems: first, single value number sentences that may be solved computationally or relationally; and two-value number sentences where the students were required to think about numbers that can vary. For these sentences, the students can use symbolic representations, but written explanation is equally acceptable.

Type I number sentences (single box) over all four operations were used where students were invited to find the value of a missing number and to explain their thinking. Examples of the problems used in the questionnaire are as follows.

$\square$	+	17	=	15	+	24
99	–	$\square$	=	90	–	59
48	×	2.5	=	$\square$	×	10
3	÷	4	=	15	÷	$\square$

Having students' answers it can be said that some students who use computation on Type I number sentences may be able to think relationally. Computational thinking may be the result of a student ***choosing*** to solve the missing number sentences by computation. Other students may use computation because that is the ***only*** method that the student is able to use. Questions are needed that require students to think relationally and to distinguish between these two groups of students.

Type II and Type III sentences are the focus of this paper. The problems consist of four (4) mathematics operations such as addition (+), subtraction (–), multiplication (x), and division (:). These problems require students to think relationally otherwise they will get confused. The following example of a Type II Sentences was given to the students (see parts *a* to *d*).

1. Can you think about the following mathematical sentence?

$$18 \quad + \quad \underset{\textit{Box A}}{\boxed{}} \quad = \quad 20 \quad + \quad \underset{\textit{Box B}}{\boxed{}}$$

(a) *In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?*

$$18 \quad + \quad \underset{\textit{Box A}}{\boxed{}} \quad = \quad 20 \quad + \quad \underset{\textit{Box B}}{\boxed{}}$$

$$18 \quad + \quad \underset{\textit{Box A}}{\boxed{}} \quad = \quad 20 \quad + \quad \underset{\textit{Box B}}{\boxed{}}$$

$$18 \quad + \quad \underset{\textit{Box A}}{\boxed{}} \quad = \quad 20 \quad + \quad \underset{\textit{Box B}}{\boxed{}}$$

(b) *When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?*

(c) *If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?*

(d) *If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.*

(e) What can you say about c and d in this mathematical sentence?

$c + 2 = d + 10$

Type III sentences, shown in part *e* above, used symbols *c* and *d* in sentences that were structurally similar to Type II Number Sentences. The Type II and III problems for subtraction, multiplication and division are contained in the appendixes 2 – 4 of this paper.

There are three categorizations of students' relational understanding based on students' answers on Type II and Type III problems.

a. Established relational thinking
b. Consolidating relational thinking
c. Emerging relational thinking

Students have established relational thinking whenever they create their responses to Type II and Type III sentences across at least three of the four operations, which demonstrate clear and correct relational thinking.

Part a	√
Part b	√
Part c	√
Part d	√
Part e	√

Findings

One of the Junior High School students (Muhammed Rizqi Musa, Year 8, clearly demonstrates established relational thinking. Students like this are able to:

– Specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the numbers, including the magnitude and direction of the difference between them.
– Employ a similar form of words used to describe this relationship as a part of the condition that describes how any number can be used in Box A and still make a true sentence.
– Explain clearly how *c* and *d* are related for the Type III sentence to be true, treating *c* and *d* as general numbers.

Some other students in their responses to Type II and Type III sentences, across at least three of the four operations, display consolidating relational thinking. These students demonstrate a clear relational thinking in parts a, b and c but have difficulty with one or both of parts d and e.

The following student, Hesti, also Year 8, shows consolidating relational thinking. Students like Hesti are almost always able to specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the numbers, including the magnitude and direction of the difference between them. In the sample of Hesti's work, shown below, we see that she is able to specify the relationship correctly between Box A and Box B in part *b*, but not in part *c*. Hesti correctly and fully specified the relationships between Box A and Box B for the other operations, but her responses to parts *d* and *e* are quite different from that of Rizqi above.

Sometimes these Consolidating students are able to refer to some feature of the relationship between *c* and *d*, or give a specific pair of values for *c* and *d*, but cannot give a complete explanation of the relationship. Hesti's response is typical of these students. We can see, however, that she is able to identify some features of the relationships in parts d and e, but is not able to identify the general relationships between Box A and Box

Other students can be seen to have an emerging relational thinking. In their responses to Type II and Type III sentences across the four operations, these students are unable to give a complete relational description in parts b and c to the relationship between the numbers in Box A and in Box B and almost always have difficulty with parts d and e.

Two Banda Aceh students, for instance: Zunnawanis (Year 8) and Putania (Year 8), are typical emerging relational thinkers. Some students like Zunnawanis typically identify a feature of the number sentences used in Box A and Box B, but do not fully specify the relationship between the numbers used in Box A and Box B. As a result of their inability to fully express the relationship between Box A and Box B in parts *b* and *c*, these students are always unable to answer successfully parts *d* and *e*.

Findings and Connections from Several Countries

Since the Australia-China study in 2008 (Stephens and Wang, 2008), this questionnaire consisting of the same Type II and Type III questions has been used in Oxford (UK), Sao Paulo (Brazil). In Banda Aceh, Indonesia of the 27 of Year 8 students who were given questionnaire, only one was an Established relational thinker, eight were classified Consolidating, and the remaining 18 were classified as Emerging. Among similar age students in all these countries, the key features which distinguish Established, Consolidating, and Emerging relational thinking are the same in every respect. However, the proportion of students in each category appears to vary between schools and between countries.

Conclusion

Most students in Banda Aceh samples still display emerging relational thinking. These emerging relational thinkers typically use Non-directed relations, Directed (no magnitude) relations and Directed (non-referenced) relations. They construct *limited* responses to questions in part b and c. These 'limited' relational descriptions seem to 'lock' students into a certain kind of thinking and *stops* them from successfully generalising answers to part d and part e. Having this conclusion, one big question can be addressed: How can teachers help students to move beyond these partial descriptions? It needs a deep understanding and broad analysis in order to create a good trajectory of learning for bridging students' emerged model of relational thinking to the established relational thinking.

Looking at the goals of the mathematics curriculum in Indonesia, where strong emphasis is placed on having students think "logically, analytically, systematically, critically and creatively"; and when students are expected to use mathematics reasoning and manipulation for generalization, it is evident that many students, even in Year 8, after starting a formal study of Algebra, are still having trouble to think algebraically about simple (Type II) number sentences and simple algebraic (Type III) expressions.

Many Consolidating Relational Thinkers are concerned that the relationships hold only for a specific range of numbers, e.g. positive whole numbers. Likewise, many of these students give one specific pair of values for the relationship between *c* and *d*. This relationship involves many subjects in mathematics such as fractions, decimal numbers, and negative numbers. It is not only in agreement with positive whole numbers (e.g. in Type II and Type III sentences). These three Types of sentences have a potential for develop algebraic thinking with respect to equivalence, attention to operations, different numbers, compensation, and generalisation.

While these *limited* relations denote an early stage of relational thinking development, teachers need to help their students to express referenced and directed relational descriptions. This may be done through highlighting to students the disadvantages and advantages that different descriptions offer.

Students should understand the value of learning algebraic sentences especially in relation to the contextual problems in their life. *Moreover*, the learning processes should cover solving different kinds of problems. Type I and Type II and Type III sentences present different ways of having students think about numbers that vary. Having these active learning experiences, students enhance and restructure their own knowledge of algebraic thinking.

Finally, there are several general recommendations to help teachers to use the potential of Type I, II, and III sentences.

- The different numerical and algebraic sentences need to be learned as a subject matter that requires a good knowledge of mathematics
- Working with and discussing several varieties of algebraic sentences in the classroom is a good experience for teachers toward the best practice of instruction.
- Guidance from experts and research in dealing with algebraic sentences, such as has been discussed in this paper and drawing on actual examples of students' work, becomes an important step towards the best practice for teaching algebraic sentences in the classroom.
-

References

-----. 2006. Peraturan Menteri Pendidikan Nasional Indonesia No. 22 Tahun 2006 tentang Standar Isi Untuk Satuan Pendidikan Dasar dan Menengah, Kurikulum Tingkat Satuan Pendidikan (KTSP). Jakarta: Depdiknas.

Carpenter, T. P., & Franke, M. L. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The Future of the Teaching and Learning of Algebra,* (Proceedings of the 12th ICMI Study Conference, pp. 155-162). Melbourne: University of Melbourne.

Fujii, T. (2003). Probing students' understanding of variables through cognitive conflict problems: Is the concept of a variable so difficult for students to understand? In N. A. Pateman, B. J. Dougherty and J. Zilliox (Eds.), *Proceedings of the Joint Meeting of PME and PMENA*. vol. 1, pp. 49–65, University of Hawai'i: PME, 2003.

Hart, K. M. (1981). *Children's understanding of Mathematics*. London: John Murray.

Irwin, K. and Britt, M. (2005). The algebraic nature of students' numerical manipulation in the New Zealand Numeracy Project. *Educational Studies in Mathematics 58*, 169-188.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12,* 317-326.

Molina, M., Castro, E., & Mason, J. (2008). Elementary students' approaches to solving true/false number sentences. *Revista de Investigacion en Didactica de la Matematica PNA 2(2)*, 75-86.

Stephens, M. (2007). "Students' emerging algebraic thinking in the middle school years". In J. Watson & K. Beswick (eds.), *Mathematics: Essential Research, Essential Practice*, (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, pp. 678-687). Hobart: MERGA.

Stephens, M. (2008). Some key junctures in relational thinking. In M. Goss, R. Brown & K. Makar (Eds.), *Navigating current and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Group of Australasia, pp. 491-498). Brisbane: MERGA.

Stephens, M., & Wang, X. (2008). Some key junctures in relational thinking. *Journal of Mathematics Education* (in Chinese), *17*(5), 36-40.

The Mathematical Association of America (2007). *Algebra: Gateway to Technological Future*, edited by V. J. Katz. MAA: author.